Chapter 1 Math Review

I. Linear Algebra.

1. Vector is an arrow: vector space. linearity.

- Addition : "swappable". $u+v = v+u$.
- Scaling . $a(u+v) = au + av$.

2. Euclidean n-d space. $\mathbb{R}^n$.

3. Functions as Vectors.

- $f+g(x) = f(x)+g(x)$. $af(x) = a \cdot f(x)$

> Ex 1 $A(a_1, b_1)$ $B(a_2, b_2)$ Midpoint
>
> Return $\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$.

4. Measuring Vectors.

need more definition.

① Norm / Magnitude / Length : size of a vector . ≥ 0 . (0 for only zero vector).

Properties : ≥ 0, $|v|=0 \Leftrightarrow v=0$, $|av| = |a||v|$. $|u+v| \geq |u|+|v|$.

- Def (L^2 Norm of f) : $\|f\| := \sqrt{\int_0^1 f(x)^2 dx}$.

② Inner Product / Dot Product

- Vectors have orientation. We want to measure how vectors "line up" with each other. → Alignment.
- Order shouldn't matter. $\langle u, v\rangle = \langle v, u\rangle$
- They can be thought as projections. $\langle u, v\rangle$ is the (norm of). projection of u onto v. A shadow of u cast on v.

Def (Inner Product). $\langle u, v\rangle = \sum_{i=1}^{n} u_i v_i$.

Def (L^2 Inner Prod. of f,g) $\langle\langle f, g\rangle\rangle := \int_0^1 f(x) g(x)\, dx$.

[Use Case] Alignment of Images. Alignment of Signals.

II. Linear Maps. f

1. Intuition : 

- Takes lines to lines. - Center Map to Center.

2. f: vector → vector. $f(u+v) = f(u) + f(v)$. $f(au) = af(u)$.

$$f(u_1, u_2, \cdots, u_m) = \sum_{i=1}^{m} u_i \vec{a}_i$$

> Ex 2 Is $f(x) = ax+b$ linear ?
>
> It's not. It's not a line go through the origin.

→ This is called Affine.

3. Span: $a\vec{u} + b\vec{v}$.

$\mathrm{span}(e_1, e_2, \cdots, e_n) = \mathbb{R}^n \Leftrightarrow e_1, e_2, \cdots, e_n$ are a basis for $\mathbb{R}^n$.

4. Gram-Schmidt Algorithm.

Given: A set of basis vectors $a_1, \dots, a_n$.

Output: An orthonormal basis $e_1, \dots, e_n$.

Algorithm: ① Normalize the first vector.

② Subtract any component of the 1st vector from the 2nd.

③ Normalize the 2^{nd} one.

④ Repeat, removing components of first k vectors from Vector $k+1$.

5. Fourier Transform.

Since functions are also vectors, they can also have ortho. basis.

Ⅱ. Matrices.

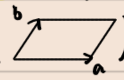
1. Encoding Linear Map.

Sps: $f(u) = u_1\vec{a}_1 + u_2\vec{a}_2$.

Encode: We want: $A := \begin{bmatrix} a_{1,x} & a_{2,x} \\ a_{1,y} & a_{2,y} \\ a_{1,z} & a_{2,z} \end{bmatrix}$.

2. Determinant for 2D Vectors.

① $|ab|$ is the (signed) area of the parallelogram formed by $\vec{a}$ and $\vec{b}$.

i.e. $\left| \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \right| = A($  $)$

3. Inverse.

$AA^{-1} = I \Leftrightarrow A^{-1}$ is the inverse of A.

⚠ $A^{-1}A = AA^{-1} = I$. $(AB)^{-1} = B^{-1}A^{-1}$

4. Transpose.

A^T is the transpose of $A \Leftrightarrow a_{ij} = a_{T\,ji}$.

5. Vector as column matrix: $v(v_1, v_2, \dots, v_m) \Leftrightarrow \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$.

Dot Product as matrix: $u \cdot v = u^T v = [u_1 \ u_2 \cdots u_m] \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$.

6. Diagonal Matrix: $\begin{bmatrix} a_1 & 0 & 0 & \cdots \\ 0 & a_2 & & \\ 0 & & \ddots & \\ \vdots & & & a_m \end{bmatrix}$ Non-zero only occurs on diagonal.

Orthogonal Matrix: each column can be considered as a normalized vector, and orthogonal to each other.
dot product = 0.

Identity: Diagonal & Orthogonal. i.e. $[1, 1, \dots, 1]$

7. Inverse Calculation: View Linear Algebra Notes.

8. Eigenvalue/Eigenvector.

Def. $Aa = \lambda a$ for matrix A and vector a. λ is call eigenvalue associated with eigenvector a.

multiply by this matrix doesn't change the direction.

Ex 3 $A=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$. Find eigenvector and eigenvalue.

The eigenvalues of A are the solutions to.

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$$\Rightarrow \lambda^2 - (2+1)\lambda + (2-1) = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 1 = 0.$$

$$\Rightarrow \lambda = \frac{3\pm\sqrt{5}}{2}$$

Now, the associated eigen vector:

$$\begin{bmatrix} 2-2.618 & 1 \\ 1 & 1-2.618 \end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = \begin{bmatrix} 0 \\ 0\end{bmatrix}.$$

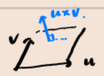
$$\begin{bmatrix} x \\ y\end{bmatrix} \approx \begin{bmatrix} 0.8507 \\ 0.8507\end{bmatrix}$$

IV. Linear Algebra [CONT.].

1. Euclidean Norm: length preserved by translation, rotation, reflections.

∴ Inner Product: $\langle u,v\rangle = |u||v|\cos\theta$.

∴ Cross Product: $u\times v$:

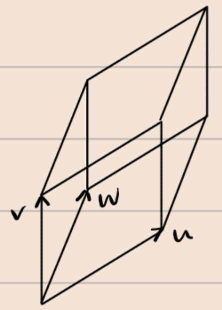


2. $u\cdot v = u^T v = [u_1 \cdots u_n]\begin{bmatrix} v_1 \\ \vdots \\ v_n\end{bmatrix} = \sum_{i=1}^{n} u_i v_i$.

3. Determinant (More).

① for $A := \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i\end{bmatrix}$. $\det A = a(ei-fh) + b(fg-di) + c(dh-eg)$.

Intuition: $\det(u,v,w)$ encodes signed volume of parallelepiped with edge vector u, v, w.



$\det(u,v,w) = (u\times v)\cdot w = (v\times w)\cdot u = (w\times u)\cdot v$.

"Triple Product".

Jacobi Identity: $u\times(v\times w) + v\times(w\times u) + w\times(u\times v) = 0$.

Lagrange Identity: $u\times(v\times w) = v(u\cdot w) - w(u\cdot v)$.

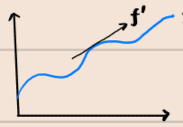
② What does it mean for $\det(A)$ for a linear map matrix A?

$f(u) = u_1 a_1 + u_2 a_2 + u_3 a_3$.

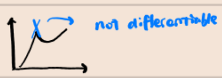
It measures the change in volume. Sign tells if orientation is flipped.

V. Differential Operators

1. Slope / Derivative: Rise over run.



differentiable if $f'^{+} = f'^{-}$.



2. Directional Derivative.

$$D_u f(x_0) = \lim_{\varepsilon \to 0} \frac{f(\vec{x}_0 + \varepsilon \vec{u}) - f(\vec{x}_0)}{\varepsilon}$$

3. Gradient: ① point to uphill, ② list of partial derivatives, ③ leads to the best possible approximation.

Def $\quad \nabla f(\vec{x}) = \begin{bmatrix} \partial f / \partial x_1 \\ \vdots \\ \partial f / \partial x_n \end{bmatrix}$ (nabla)

Ex 4 $\quad f(\vec{x}) := x_1^2 + x_2^2$.

$$\frac{\partial f}{\partial x_1} = 2x_1 + 0. \qquad \frac{\partial f}{\partial x_2} = 0 + 2x_2$$

$$\Rightarrow \nabla f(\vec{x}) = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = 2\vec{x}.$$

⚠ Increase value of f as quick as possible.

4. Gradient & Directional Derivative.

$\langle \nabla f(x), \vec{u} \rangle = D_u f(x). \quad \forall u$ (gradient) Must be differentiable.

5. L^2 Gradient for function F.

$\langle\langle \nabla F, u \rangle\rangle = D_u F$

VI. Vector Fields/ Operators.

1. Measure the change in a Vector Field. X.

div X (how much shrinking /expanding): $\nabla \cdot X = \left(\frac{\partial x_1}{x_1}, \frac{\partial x_2}{x_2}, \cdots, \frac{\partial x_n}{x_n} \right)$.

curl X (how much spinning (clockwise)). $\nabla \times X = \begin{bmatrix} \partial x_3/\partial u_2 - \partial x_2/\partial u_3 \\ \partial x_1/\partial u_3 - \partial x_3/\partial u_1 \\ \partial x_2/\partial u_1 - \partial x_1/\partial u_2 \end{bmatrix}$

2. Laplacian. Δ

$$\Delta f := \nabla \cdot \nabla f = \text{div}(\text{grad } f)$$

$$:= \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

$$:= -\nabla_f \left(\tfrac{1}{2} \|\nabla f\|^2 \right).$$

Ex 5 $\quad f(x_1, x_2) := \cos(3x_1) + \sin(3x_2)$.

$$\Delta f = \frac{\partial^2}{\partial^2 x_1} f + \frac{\partial^2}{\partial^2 x_2} f = -9(\cos(3x_1) + \sin(3x_2))$$

3. Hessian. ∇^2

$$(\nabla^2 f)\, \vec{u} := D_{\vec{u}}(\nabla f)$$