Chapter 2 Basic Transform

I. Linear Transformations.

1. 2D Scale.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}}_{S_a} \begin{bmatrix} x \\ y \end{bmatrix} \quad \xRightarrow{\text{Homo}} \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

2. 2D Rotation. (θ rotate around $\vec{0}$).

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{R_\theta} \begin{bmatrix} x \\ y \end{bmatrix} \quad \xRightarrow{\text{Homo}} \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3. 2D Shear - x

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & sh_x \\ 0 & 1 \end{bmatrix}}_{H_{xs}} \begin{bmatrix} x \\ y \end{bmatrix}. \quad \xRightarrow{\text{Homo}} \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Shear - y.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ sh_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \xRightarrow{\text{Homo}} \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

4. 3D Scale.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & S_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \xRightarrow{\text{Homo}} \quad \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

5. 3D Shear. (Along x).

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & d_y & d_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \xRightarrow{\text{Homo}} \quad \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & d_y & d_z & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}.$$

II. Affine Transformation.

1. Def (Homogeneous Coordinates) Add an additional $(d+1)$th dimension w to a point or vector. in $\mathbb{R}^d$.

such that :

- if $\vec{P} = (x_1, x_2, \cdots, x_d)$ is a point, then $\vec{P}_H = (x_1, x_2, \cdots, x_d, 1)$.
- if $\vec{P} = (x_1, x_2, \cdots, x_d)$ is a vector, then $\vec{P}_H = (x_1, x_2, \cdots, x_d, 0)$.

$\vec{P}_H$ is called the homogeneous coordinate of $\vec{P}$.

⚠ Why do we need HC ?

① Tell if $\vec{P}$ is a vector or a point.

② Turn Affine Transformations into Linear.

2. 2D Translation by $\vec{b} = (t_x, t_y)$.

$x' = x + t_x$. $\quad y' = y + t_y \quad \overset{\text{Homo}}{\Rightarrow} \quad \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

3. 3D Translation by $\vec{b} = (t_x, t_y, t_z)$

$$\begin{cases} x' = x + t_x \\ y' = y + t_y \\ z' = z + t_z \end{cases} \quad \overset{\text{Homo}}{\Rightarrow} \quad \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

III. Rotation Analysis

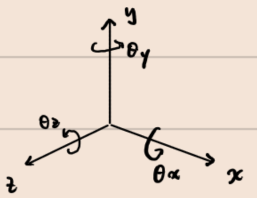
1. 3D Rotation is not communitive

- 2D Rotation: Rotate 20° then 40° $\Leftrightarrow$ Rotate 40° then 20°.
- 3D Rotation: Rotate 20° around x then 90° around y

  $\not\Leftrightarrow$ Rotate 90° around y then 20° around x.

2. Representation.

① Euler Angle.



Determine 3D rotation by rotations around 3 axises.

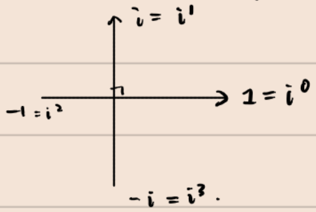
May have <u>Gimbal Lock</u>: Let $\theta_y = \pi/2$ (i.e. $\cos\theta_y = 0$, $\sin\theta_y = 1$).

No matter how we change θ_x, θ_z,

rotations happen in only one plane.

② Complex Number

Let (r, θ) represent $z = a + bi$, where $r = \sqrt{a^2 + b^2}$, $\theta = \arctan\left(\frac{b}{a}\right)$.



Now we will have $\begin{matrix} z_1 = (r_1, \theta_1) \\ z_2 = (r_2, \theta_2) \end{matrix} \Rightarrow z_1 z_2 = (r_1 r_2, \theta_1 + \theta_2)$.

Ex Show the above $z_1 z_2 = (r_1 r_2, \theta_1 + \theta_2)$

Let $z_1 = a_1 + b_1 i$. $\quad z_2 = a_2 + b_2 i$.

$z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i)$

$= a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2$

$= a_1 a_2 - b_1 b_2 + (a_1 b_2 + a_2 b_1) i$

Now, for $z_1 z_2$,

$R = \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2}$

$= \sqrt{a_1^2 a_2^2 + b_1^2 b_2^2 - \cancel{2 a_1 a_2 b_1 b_2} + a_1^2 b_2^2 + a_2^2 b_1^2 + \cancel{2 a_1 a_2 b_1 b_2}}$

$= \sqrt{(a_1 a_2)^2 + (b_1 b_2)^2 + (a_1 b_2)^2 + (a_2 b_1)^2}$

$r_1 r_2 = \sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2} = \sqrt{(a_1^2 + b_1^2)\cdot(a_2^2 + b_2^2)} = \sqrt{a_1^2 a_2^2 + a_1^2 b_2^2 + b_1^2 a_2^2 + b_1^2 b_2^2} = R$.

$$\theta = \arctan\left(\frac{a_1b_2 + a_2b_1}{a_1a_2 - b_1b_2}\right)$$

$$\theta_1 = \arctan\left(\frac{b_1}{a_1}\right) \qquad \theta_2 = \arctan\left(\frac{b_2}{a_2}\right).$$

$$\begin{aligned}\theta_1 + \theta_2 &= \arctan\left(\frac{b_1}{a_1}\right) + \arctan\left(\frac{b_2}{a_2}\right)\\ &= \arctan\left(\frac{b_1/a_1 + b_2/a_2}{1 - b_1b_2/a_1a_2}\right)\\ &= \arctan\left(\frac{b_1a_2 + b_2a_1}{a_1a_2 - b_1b_2}\right) = \theta.\end{aligned}$$

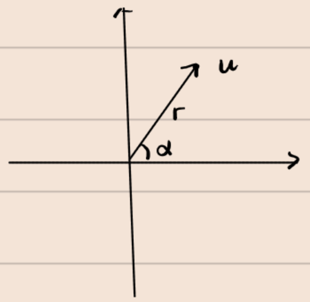
Now, Euler Function. $e^{i\theta} = \cos\theta + i\sin\theta$. Consider $\overset{z_1}{(r=a,\theta)}$ and $\overset{z_2}{(r=a,\varphi)}$

For $z_1 = a\cos\theta + a\sin\theta\, i = a(\cos\theta + i\sin\theta) = ae^{i\theta}$.

$z_2 = ae^{i\varphi}$. $\Rightarrow z_1z_2 = abe^{i(\theta+\varphi)}$

In this way, we can say that z_1 indicates a vector whose magnitude is a with a rotation θ. z_2, b, φ. and z_1z_2, ab, $\theta+\varphi$

Polar / Complex



For a vector $u = re^{i\alpha}$.

- Rotation $a = e^{i\theta}$. $b = e^{i\varphi}$.

$$A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \qquad B = \begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}.$$

- Apply a θ rotation $\Rightarrow$ $au = re^{i(\alpha+\theta)}$
  $\Updownarrow$
  $Au = A\begin{bmatrix}x\\ y\end{bmatrix}$.
- Apply a $\theta+\varphi$ rotation $\Rightarrow$ $abu = re^{i(\alpha+\varphi+\theta)}$.

③ Quaternion.

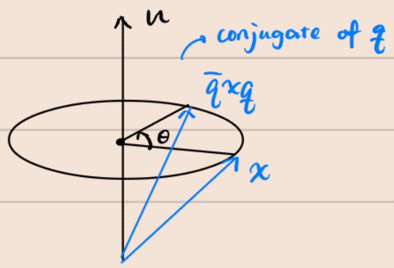
<u>Def</u> $\mathbb{H} := \text{span}\{i, j, k, 1\}$, i.e. $\forall q = a + bi + cj + dk$, $q \in \mathbb{H}$.

<u>Prop</u> $i^2 = j^2 = k^2 = ijk = -1$

<u>Encode</u> $(x, y, z) \mapsto 0 + xi + yj + zk$

<u>Representation</u> Given axis $\vec{u}$, angle θ, quaternion q representing:

$$q = \cos(\theta/2) + \sin(\theta/2)\, u.$$



Good for interpolation.

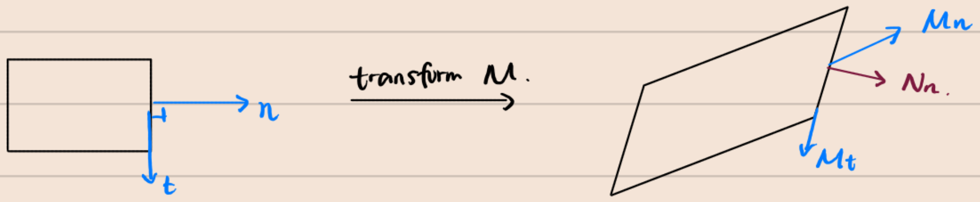
IV. Composition / Decomposition of Transformation

1. First apply scale S, then rotation R. $\Rightarrow v' = RSv$. ← Left Multiplication.

⚠ Note that in general. $RSv \neq SRv$.

2. Transforming Normal Vectors

In general. normal vectors are not preserved after transformation. M.



We need to set normal after transforming (denoted by N).

We know that ① $n \cdot t = 0$ i.e. $n^T t = 0$

② Let $n_N = \underbrace{Nn}_{\text{this is transform } N \text{ multiplying vector } n.}$ $t_M = Mt$

We know that $n_N \cdot t_M = 0$. i.e. $n_N^T t_M = 0$.

Algebraic calculation.

$$n^T t = n^T \underbrace{I}_{\text{Identity}} t$$

$$= n^T M^{-1} M t$$

$$= (n^T M^{-1})\, t_M$$

$$= 0$$

$$\Rightarrow n^T M^{-1} = n_N^T$$

$$\Rightarrow n_N = (n^T M^{-1})^T = (M^{-1})^T n.$$

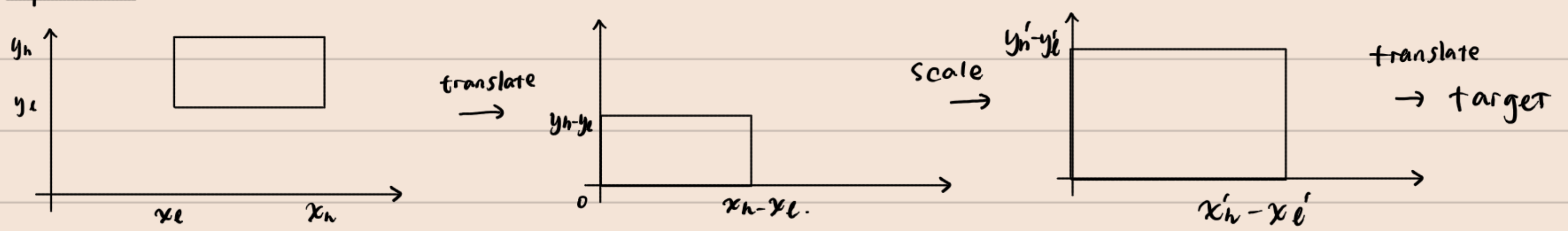
$$\Rightarrow N = (M^{-1})^T$$

The transpose of the inverse matrix preserves the normal.

3. Windowing Transformation.

Create a transform matrix that transforms $(x, y) \in [x_\ell, x_h] \times [y_\ell, y_h]$ to $[x'_\ell, x'_h] \times [y'_\ell, y'_h]$

Operation



Thus. Window = Translate (x'_ℓ, y'_ℓ) Scale $\left(\frac{x'_h - x'_\ell}{x_h - x_\ell}, \frac{y'_h - y'_\ell}{y_h - y_\ell}\right)$ · translate $(-x_\ell, -y_\ell)$